The Category of Sets in Homotopy Type Theory

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What is a set in HoTT?

A set in HoTT is a type in which there is at most 1 path between any two points in a space. That is, given any 2 points x, y : A, and any two paths p, q: x = y, then p = q

$$\mathsf{isSet}(A) :\equiv \prod_{(x,y:A)} \prod_{(p,q:x=y)} (p = q)$$

The type (or category) of sets is then the space which has all types which are sets as objects. As an example in Figure 1,, the universe is not a set.

$$\mathbf{Set} :\equiv \sum_{A:\mathcal{U}} \mathsf{isSet}(A)$$

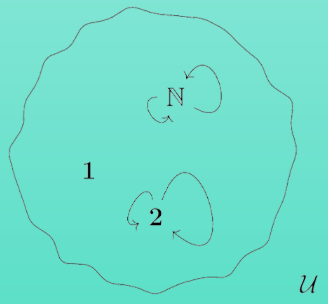

Figure 1. The universe is not a set

The category of sets

Set forms a ΠW-pretopos, i.e.:

1. Set is <u>regular</u>: the following is a pullback square, with *m* injective and *n* surjective:

$$\begin{tikzcd}
A \times_C D \arrow[r, "n'"] \arrow[d, "\pi_1"'] \arrow[rr, bend left, "\pi_2"] & B \times_C D \arrow[r, "m'"] \arrow[d, "\pi_1'"] & D \arrow[d, "\pi_1''"] \\
A \arrow[r, "n"'] \arrow[rr, bend right, "f"'] & B \arrow[r, "m"'] & C
\end{tikzcd}$$

2. Set is <u>exact</u>: the following is a pullback square, a ~ b ≃ (q(a) = q(b)), where "~" is an equivalence relation

$$\begin{tikzcd}
\sim \arrow[r, "r_2"] \arrow[d, "r_1"'] & S \arrow[d, "q"] \\
S \arrow[r, "q"'] & S/\sim
\end{tikzcd}$$

3. Set is a <u>pretopos</u>, i.e, the following holds:

$$(A \times_D C) + (B \times_D C) \simeq (A + B) \times_D C$$

4. Set is a <u>ΠW-pretopos</u>: i.e., it is a locally cartesian category, with W-types

Comparing HoTT and ZFC

Foundational differences

Primitives of the theory:

1. HoTT has types
2. ZFC has sets and the membership relation "∈"

The novelty in HoTT comes in the addition of the Univalence Axiom:

$$(A =_{\mathcal{U}} B) \simeq (A \simeq_{\mathcal{U}} B)$$

However, (ZFC + Univalence) is inconsistent, so we cannot simply add the axiom to ZFC.

Three ZFC axioms in HoTT

The Empty Set axiom in ZFC is a theorem in HoTT

$$\exists x \forall y \neg (y \in x) \qquad \frac{\Gamma \ \mathsf{ctx}}{\Gamma \vdash \mathbf{0} : \mathcal{U}_i} \text{ 0-form}$$

The Power Set Axiom is not present in HoTT without the propositional resizing axiom, which says that the proposition types are equivalent for all homotopical levels, which would make HoTT impredicative

$$\forall x \exists y \forall z (z \in y \iff \forall w (w \in z \to w \in x))$$

Due to its inherently non-constructive nature, it is surprising that HoTT does exhibit the axiom of choice. Further, since AC implies LEM, one can model ZFC in HoTT

$$\left(\prod_{x:X} \left(\left\| \sum_{a:A(x)} P(x,a) \right\| \right) \right) \to \left\| \sum_{(g:\prod_{(x:X)} A(x))} \prod_{(x:X)} P(x, g(a)) \right\|$$