Formalising the Symmetry Book

Goal of the project

The goal of the project is to formalise all the proofs in chapter 3 (*The universal symmetry: the circle*) of the Symmetry Book[1] in the UniMath library, so its correctness can be verified and the theorems/lemmas/colloraries can be used in other proofs. The research question is: *Can we formalise the proofs in chapter 3 of the Symmetry book using the UniMath Coq library to verify their correctness?*

The Symmetry Book

The Symmetry book is a bachelor's level textbook which teaches about the symmetries in mathematics from the perspective of homotopy type theory.

Homotopy Type Theory

Homotopy type theory[2] is a dependent type theory which considers types to be topological spaces. Furthermore, equality between points is given by a path between points within a topological space. An example of a space with two equal points is denoted in figure 1.

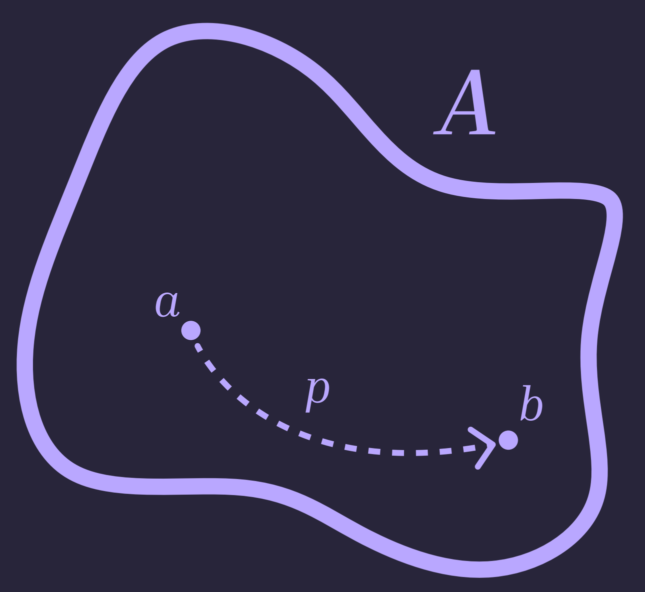

figure 1. two equal elements of a type.

The Circle

The circle is a higher inductive type. Figure 3 shows the circle. The circle has two constructors, namely `base : circle`, which is a point on the circle, and the path from the point, all around the circle, to itself, that is `loop : base` $=_{\texttt{circle}}$ `base`.

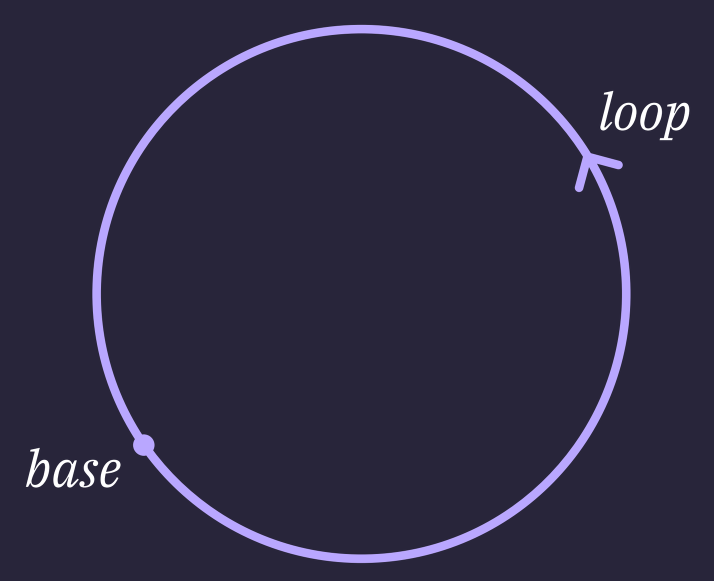

figure 2. a circle.

Formalised Proofs

Below is a list of all the statements I proved using Coq UniMath.

Lemma 5.1. *The circle is connected.*

$$\prod_{z:S^1} \|\bullet \overset{=}{\to} z\|$$

Theorem 5.2. *For all types A, the evaluation function*

$$\mathtt{ev}_A : (S^1 \to A) \to \sum_{a:A}(a \overset{=}{\to} a)$$

defined as $\mathtt{ev}_A(g) :\equiv (g(\bullet), \mathtt{ap}_g(\circlearrowleft))$ *is an is an equivalence.*

Construction 5.3. *We have the following equivalence.*

$$(A \to Set) \xrightarrow{\simeq} \sum_{B:\mathcal{U}} \sum_{f:B\to A} \prod_{a:A} isSet(f^{-1}(a))$$

Theorem 5.4. *The evaluation function provides an equivalence*

$$\mathtt{ev}_{Set} : (S^1 \to Set) \to \sum_{X:Set}(X \overset{=}{\to} X)$$

defined by $\mathtt{ev}_{Set}(g) :\equiv (g(\bullet), ap_g(\circlearrowleft))$

Consequently, we have a string of equivalences

$$\begin{aligned} SetBundle(S^1) &\xrightarrow{\simeq} (S^1 \to Set) \\ &\xrightarrow{\simeq} \sum_{X:Set}(X \overset{=}{\to} X) \xrightarrow{\simeq} \sum_{X:Set}(X \xrightarrow{\simeq} X) \\ &\xrightarrow{\simeq} \sum_{X:U}\sum_{f:X\to X} isSet(X) \times isEquiv(f) \end{aligned}$$

Use of AI tools

To assist to me in writing my proofs, I tried using both Github Copilot X and ChatGPT. Only Github Copilot X proved to be useful, and that only for small steps inside the proof. This did not make the work much faster, as most of the time was spent with larger, more complicated steps, but was sometimes helpful.

Conclusion & Future Work

We formalised four proofs in the end, of which two theorems. Although we could not formalise all of chapter 3 nor all of the sections due to time constraints, all the attempted proofs were succesfully formalised.

In future research, more of the Symmetry book should be formalised. As the Symmetry book is a work in progress, the collection of formalisations should grow as it grows.

Project

Computer Checked Proofs for Homotopy Type Theory

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Footnotes

1. Bezem, Marc, Ulrik Buchholtz, Pierre Cagne, Bjørn Ian Dundas, and Daniel R. Grayson. 'Symmetry', 10 May 2023. https://github.com/UniMath/SymmetryBook.
2. Awodey, Steve, Thierry Coquand, and Vladimir Voevodsky. Homotopy Type Theory: Univalent Foundations of Mathematics. Univalent Foundations Program, Institute for Advanced Study, 2013.

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