Kuratowski Finite Sets in the UniMath library

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Background

- Homotopy Type Theory is a new field of Mathematics that at its core uses types. These types can be compared with the each other using the Univalence axiom [1].
- As type theory reasons about types, a way to reason about sets has to be implemented separately within Homotopy Type Theory. One of the ways you can implement it is using Kuratowski Finite Sets as proposed by Frumin et al [2].
- In Homotopy Type Theory there is a prevalent library, namely the UniMath library for which Kuratowski Finite Sets are not defined [3]. This library could however use these definitions.

Research Question

Can the proofs given in Finite Sets in Homotopy Type Theory be verified for correctness in the UniMath library [2, 3]?

Methodology

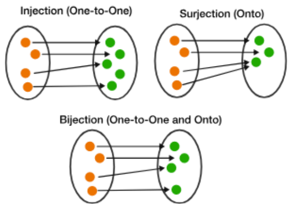
The research was split up into to different parts.

- Learning about UniMath, Homotopy Type Theory, Coq and Finite Sets.
- Defining and proving statements that were defined and proven in Finite Sets in Homotopy Type Theory [2].

Related works

To understand my research, it is important to first know a few of the key concepts of Homotopy Type Theory and finite sets.

- A set is Kuratowski finite is there exist a surjection from a set of finite natural numbers onto it. This contrasts with the more generally know Bishop finite in which case there has to be a bijection instead of a surjection.



- It is important to know what a proposition is and what a set is. A type is a proposition if for all its witnesses, the witnesses are equal. Forall (x y : A), x = y.
- A type is a set if its witnesses are equal, the equality is a proposition. Forall (x y : A), is a prop(x = y).
- Higher inductive types are also useful, they allow for defining a type by giving inductive constructors and/or equations for that type. An example of such a type is the truncation type, this makes a new type which has either 0 or 1 element.

$$\begin{aligned}&\|A\| := \\ &\mid tr : A \to \|A\| \\ &\mid trc : \Pi(x, y : \|A\|), x = y\end{aligned}$$

Results

I have defined a Higher Inductive Kuratowski type, of which its elements are Kuratowski finite. It first 3 constructors are point constructor. Which define how you can add elements in the Kuratowski finite. The other constructors are path constructors and they are the "rules" of the finite.

$$\begin{aligned}&\mathcal{K}(A) := \\ &\mid \emptyset : \mathcal{K}(A) \\ &\mid \{\cdot\} : A \to \mathcal{K}(A) \\ &\mid \cup : \mathcal{K}(A) \to \mathcal{K}(A) \to \mathcal{K}(A) \\ &\mid idem : \Pi(x : \mathcal{K}(A)), \{x\} \cup \{x\} = x \\ &\mid nl : \Pi(x : \mathcal{K}(A)), \emptyset \cup x = x \\ &\mid nr : \Pi(x : \mathcal{K}(A)), x \cup \emptyset = x \\ &\mid assoc : \Pi(x, y, z : \mathcal{K}(A)), x \cup (y \cup z) = (x \cup y) \cup z \\ &\mid comm : \Pi(x, y : \mathcal{K}(A)), x \cup y = y \cup x \\ &\mid trunc : isaset(\mathcal{K}(A))\end{aligned}$$

I have proven with this definition of the Kuratowski type that x ∪ x = x.

I have also defined subset.

$$\begin{aligned}&\emptyset \subseteq a \equiv htrue \\ &\{b\} \subseteq a \equiv b \in a \\ &\{x, y\} \subseteq (a \cup a) \equiv x \subseteq a \wedge y \subset a\end{aligned}$$

I have also defined membership.

$$\begin{aligned}&a \in \emptyset \equiv hfalse \\ &a \in \{b\} \equiv \|a = b\| \\ &a \in (x \cup y) \equiv a \in x \vee a \in y\end{aligned}$$

I have proven for these types that they follow for the Kuratowski type.

Conclusion

I have shown that the proofs and definition given in Finite Sets in Homotopy Type Theory up to membership and subset can be implemented and sufficiently reasoned about within the UniMath library [2, 3]. I have shown that the Kuratowski type can be defined, its recursion principle, induction principle and induction property can be defined and that you can reason about relationships of this Kuratowski type.

I have also shown that you can define subset and membership of the Kuratowski type so you can reason on the members of the Kuratowski finite sets.

Future Work

There is still a lot of work that could be done working on finite sets. It might be useful to finish the proofs on extensionality based on membership and subset of the Kuratowski type.

It might also be useful to create a listed set of which the size is known and proof an equality between the listed set and the Kuratowski type. This is useful as you get a way to reason about the exact size of the Kuratowski finite set.

In a similar fashion one could proof extensionality with booleans instead of using hProps, allowing for a different reasoning of the Kuratowski finite set.

One could also implement finite sets using a different kind of finite such as Bishop finite as that would allow for different ways to describe finite sets in Homotopy Type Theory.

References

1. The Univalent Foundations Program. Homotopy type theory: Univalent foundations of mathematics. arXiv preprint arXiv:1308.0729, 2013
2. Frumin, Geuvers, Gondelman van der Weide. Finite sets in homotopy type theory. CPP 2018: Proceedings of the 7th ACM SIGPLAN International Conference on Certified Programs and Proofs, pages 201–214, 1 2018.
3. Vladimir Voevodsky, Benedikt Ahrens, Daniel Grayson, et al. Unimath — a computer-checked library of univalent mathematics. available at http://unimath.org.