A Comparison of Rigour and Intuition in Illustrations in Calculus and Analysis Textbooks

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Background

- Visualisations in mathematics are not only important tools to aid in understanding, but they form a source of knowledge, e.g. they help you prove a conjecture [1]
- In textbooks, they can serve different roles
- Depending on the context, they support intuition and demonstrate some procedure [2] or they guide reasoning without replacing formal arguments [3]

- Studies have analysed the differences between these books [4] [5]
- In particular, there are different messages in pre-calculus, calculus and analysis textbooks [6]

- The research gap is that there has been no study into the differences in epistemic roles of illustrations between calculus and analysis book with attention to **rigour** and **intuition**

Research Question

- How are illustrations used and what communicative functions do they serve in computation-oriented calculus versus proof-oriented analysis textbooks?

Methodology

- Iterative open coding and thematic analysis => 129 figures
- Topic selected: Continuity, Differentiation & Integration
- Books selected: Introduction to Analysis by Jan van Neerven, An introduction to Proof by Steven R. Lay and Early Transcendentals by James Stewart
- Three main code families used: Definition, Example and Proof
- Codes used in these families:
- Definition: *formal definition, informal definition, introduction*
- Example: *counterexample, confirming example, intuition-building example, exercise*
- Proof: *proof by picture, outline for the proof, statement*

Results & Conclusion

The main theme is: illustrations are adapted or even re-contextualized in the analysis textbook and in the calculus book to fit the focus of rigour and the focus of intuition, respectively

This theme will be subdivided into subthemes with their own patterns yet all connected through the main theme: definitions, examples and proofs. The differences in the illustrations that underpin these subthemes are supported by the figures given below.

The subtheme regarding definitions is: illustrations associated with definitions are more frequently used in calculus textbooks to motivate and introduce a concept or illustrate an informal definition, whereas analysis textbooks overwhelmingly employ illustrations as a visual representation for formal definitions.

The subtheme regarding examples is: The epistemic role of illustrations for the examples varies. Example-based learning is mainly supported in calculus textbooks. On the other hand, the validity of mathematical statements is to a greater degree challenged in analysis textbooks.

The subtheme regarding proofs is: In calculus books illustrations sometimes take the role of a proof, whereas analysis books employ illustrations to demonstrate the main idea of a proof.

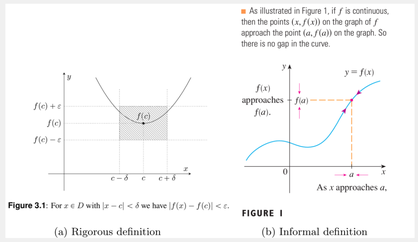
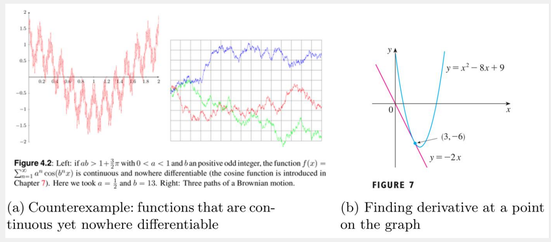


Figure in Van Neerven's book that illustrates a *formal definition* of continuity (on the left) and Figure in Stewart's book that illustrates an *informal definition* of continuity (on the right)



Figures in Van Neerven's book that illustrate *counterexamples* to the statement 'Continuous functions are also differentiable. (on the left) and Figure in Stewart's book that is a *confirming example* of the statement that the slope of the tangent line is equal to the derivative of the function at that point (on the right)

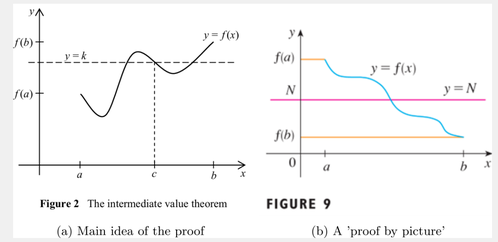


Figure in Lay's book that illustrates the *outline for the proof* of the intermediate value theorem (on the left) and Figure in Stewart's book that gives a *proof by picture* (on the right)

Discussion & Future Work

- Illustrations are not merely decorations, but are there to help the reader build intuition or visually represent a part of a mathematical exposition without losing much rigour.
- Terence Tao: *pre-rigorous stage – rigorous stage – post-rigorous stage*, where in the last stage you lead with intuition [7]
- Rigour and Intuition are not mutually exclusive; rigour is a tool to place your 'good' intuition on firmer foundation

- Effect of illustrations on (in)correct understanding of infinite dimensional spaces in advanced analysis by showing the finite dimensional pictures

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