# The Proof of the Fundamental Group of the Circle in Homotopy Type Theory's Dependence on the Univalence Axiom

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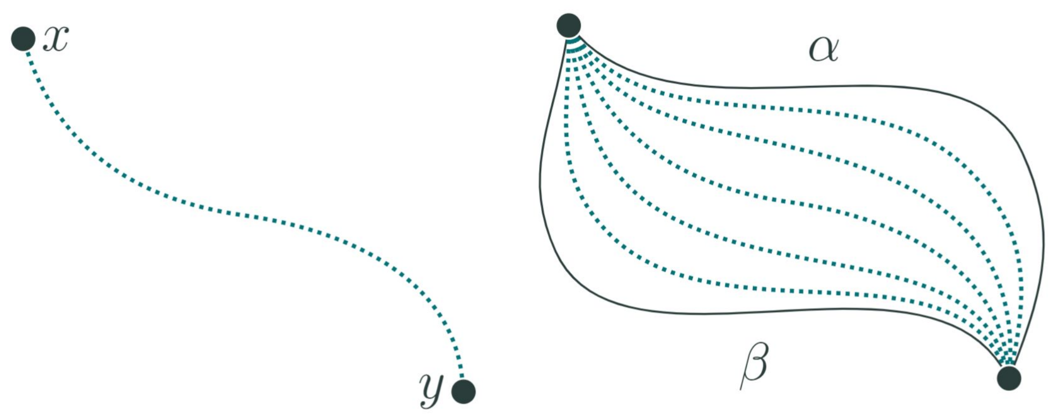
## 1 Homotopy Theory

- Mathematical context
- Homotopies are continuous maps
- Homotopies exist between points and paths
- Isomorphic paths are equal up to homotopy

*Figure 1: Visualisation of a homotopy between points x and y (left) and a homotopy between paths α and β(right)*




## 2 Homotopy Type Theory (HoTT)

- Mix between type theory, homotopy theory and category theory
- Simplifies some existing proofs
- Facilitates development of new proofs
- Tokens and types can be interpreted as
  - Points and spaces (Homotopy theory)
  - Objects and categories (Category theory)
  - Proofs and propositions (Curry-Howard isomorphism)

## 3 The Univalence Axiom (UA)

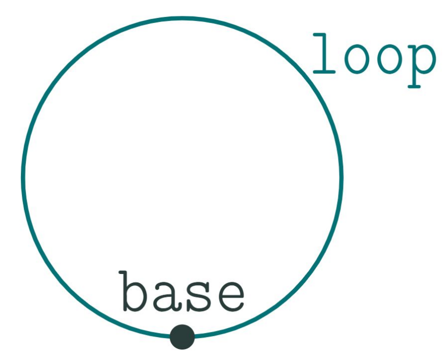
- Addition to HoTT by Voevodsky
- Creates a universe of types where equivalence can be mapped to equality
- Considers that mapping an equivalence:

$$(A \simeq B) \simeq (A =_{\mathcal{U}} B)$$

## 4 Axiom K

- Consistent to assume without univalence
- All identifications (equalities) are trivial self-identifications (reflexivity)

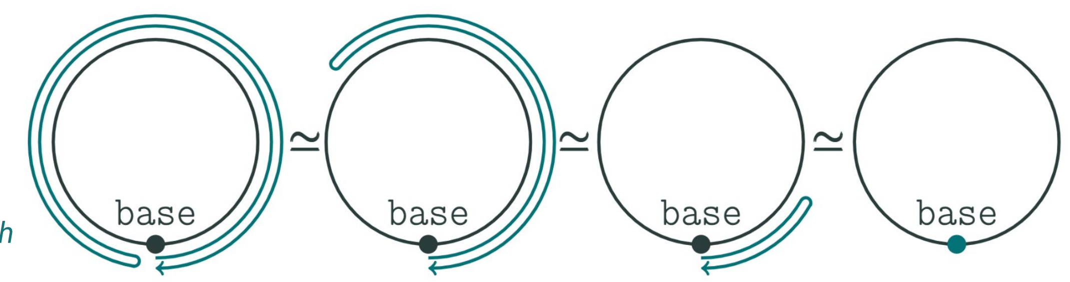
*Figure 2: Visualisation of the circle with its two constructors base and loop*




## 5 The Circle

- Higher-inductive type with 2 constructors
  - A point **base**
  - A non-trivial path **loop**
- Additional paths can be constructed by concatenation (∘) and inversion (**!**)
- E.g. **loop∘loop** or **!loop**

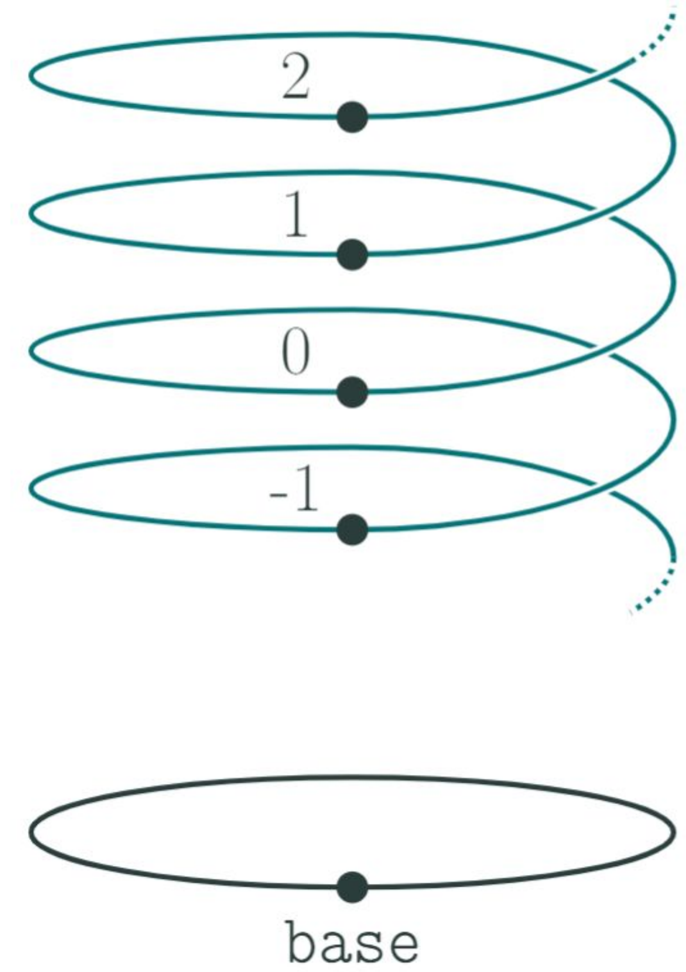
*Figure 3: Visualisation of morphing two concatenated inverse paths to the constant path*




## 6 The Fundamental Group of the Circle

- Algebraic invariant that describes sets of paths that are equal up to homotopy
- Circle's fundamental group is the integers
- Proof given by [1]
- Map paths on circle to paths on helix and label possible endpoints on helix with integers
- Visualised in Figure 4
- Map paths to helix to integers by mapping:
  - **base** to **0** to **0**
  - **loop** to **moving up** to **+1**
  - **!loop** to **moving down** to **-1**
- **+1** and **-1** are inverse like **loop** and **!loop**
- Essentially computes winding number

*Figure 4: Visualisation of correspondence between the circle and the integers on its universal cover (the helix). Adapted from [2]*




## 7 Conclusion

- **loop** must be non-trivial
- Without UA we may assume axiom K
- Under axiom K **loop** must be trivial
- Circle can not be constructed as before
- Approximation of the circle uses constant path for **loop**
- Concatenation and inversion of constant path results in constant path
- Only one path possible
- Fundamental group no longer integers but **1**

## References

[1] D. R. Licata and M. Shulman, "Calculating the fundamental group of the circle in homotopy type theory," arXiv.org, 1 2013. [Online]. Available: https://arxiv.org/abs/1301.3443
[2] T. Univalent Foundations Program, Homotopy Type Theory: Univalent Foundations of Mathematics. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.