Inverse Reconstruction with Physics-Informed Neural Operators for Darcy Flow

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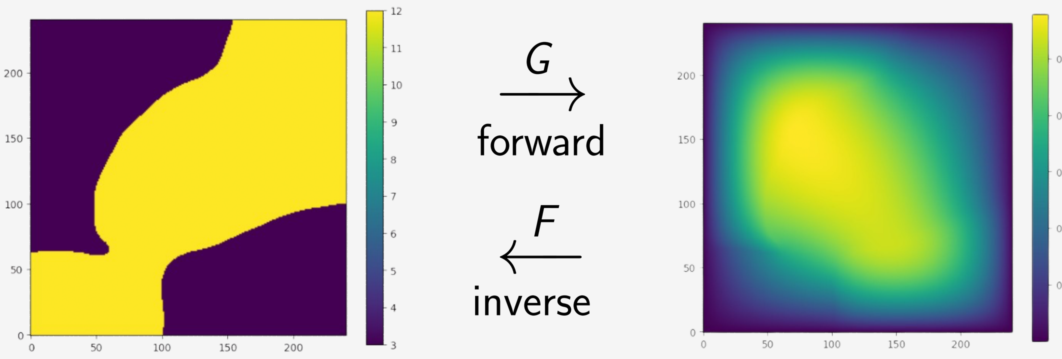
Motivation: Inverse Darcy Reconstruction 1

Inverse PDE problems aim to recover hidden physical parameters from observed solution fields.

In this project, the goal is to reconstruct the unknown permeability field $a(x)$ from pressure observations $u(x)$ in steady-state Darcy flow:

$$-\nabla \cdot (a(x)\nabla u(x)) = f(x), \qquad u|_{\partial D} = 0.$$



Permeability field $a(x)$ Pressure field $u(x)$

Research Question 2

Question: How does a PDE residual loss affect the **accuracy**, **physical consistency**, and **robustness** of inverse Darcy permeability reconstruction from noisy and sparse pressure observations using neural operators?

Neural Operators 3

Neural operators learn mappings between entire fields:

$$F_\theta : u(x) \mapsto \hat{a}(x).$$

After training, the inverse operator reconstructs the full permeability field in a single forward pass.

FNO learns this inverse map from paired data. PINO uses the same backbone, but adds a PDE residual penalty:

$$\mathcal{L}_{\mathrm{PINO}} = \mathcal{L}_{\mathrm{data}} + \lambda_{\mathrm{PDE}}\mathcal{L}_{\mathrm{PDE}} + \lambda_{\mathrm{TV}}\mathcal{L}_{\mathrm{TV}}$$

$$\mathcal{L}_{\mathrm{PDE}} = \|-\nabla \cdot (\hat{a}\nabla u) - f\|_2^2.$$

Sparse Embedding 5

Sparse observations are encoded by a binary mask M_u, where observed grid points are set to 1 and missing points to 0:

$$u_{\mathrm{obs}} = M_u \odot u.$$

The mask is provided as a second input channel, so the model can distinguish missing values from true zero pressure values:

$$F_\theta : (u_{\mathrm{obs}}, M_u) \mapsto \hat{a}$$

Training Variants 4

Variant	Role
FNO	Data-driven baseline
PINO	Adds PDE residual regularization
Physics-only	Tests PDE loss without supervision
Noise-aug.	Noise augmented training input
Sparse-trained	Tests mask-aware reconstruction

Experimental Design 6

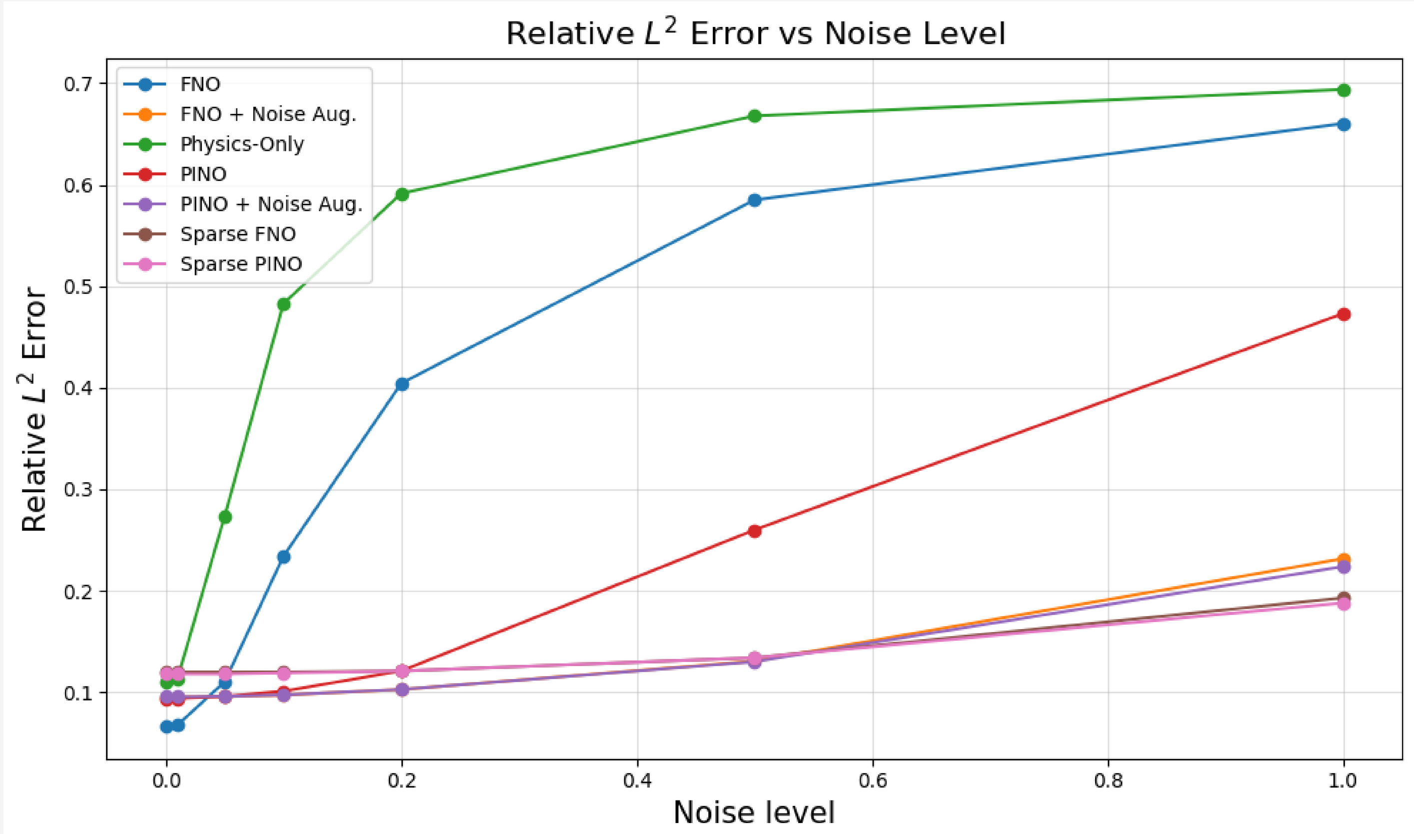
Condition	Purpose
Clean full-field u	Baseline reconstruction accuracy
Noisy full-field $\tilde{u}$	Robustness to measurement noise
Sparse ($M_u \odot u$, M_u)	Reconstruction from partial observations
Sparse and noisy ($M_u \odot \tilde{u}$, M_u)	Combined degradation setting

Clean Full-Field Baseline 7

Model	Rel. L^2	PDE	Corr.
FNO	**0.066**	0.243	**0.989**
PINO	0.094	0.317	0.979
Physics-only	0.112	**0.126**	0.980

Interpretation: Lower reconstruction error and lower PDE residual are not the same objective. The physics-only model satisfies the PDE most strongly, but does not recover the true permeability most accurately.

Main Result: Robustness Under Noise 8

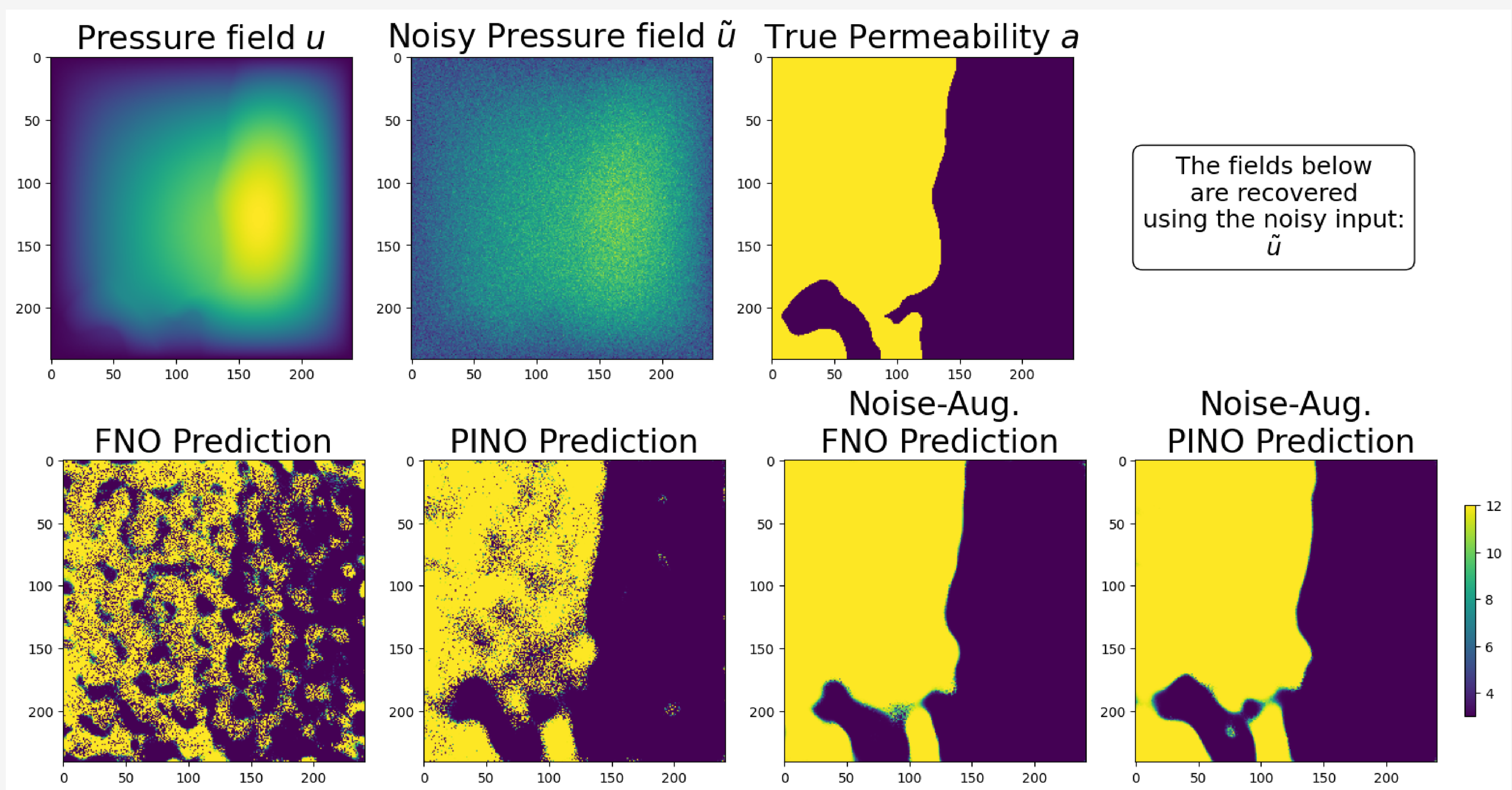


Key observation: clean-trained FNO gives the best clean-input accuracy, but degrades rapidly under noise. PINO is more stable, while noise augmentation and sparse training give the strongest robustness.

Noisy Full-Field Reconstruction 9

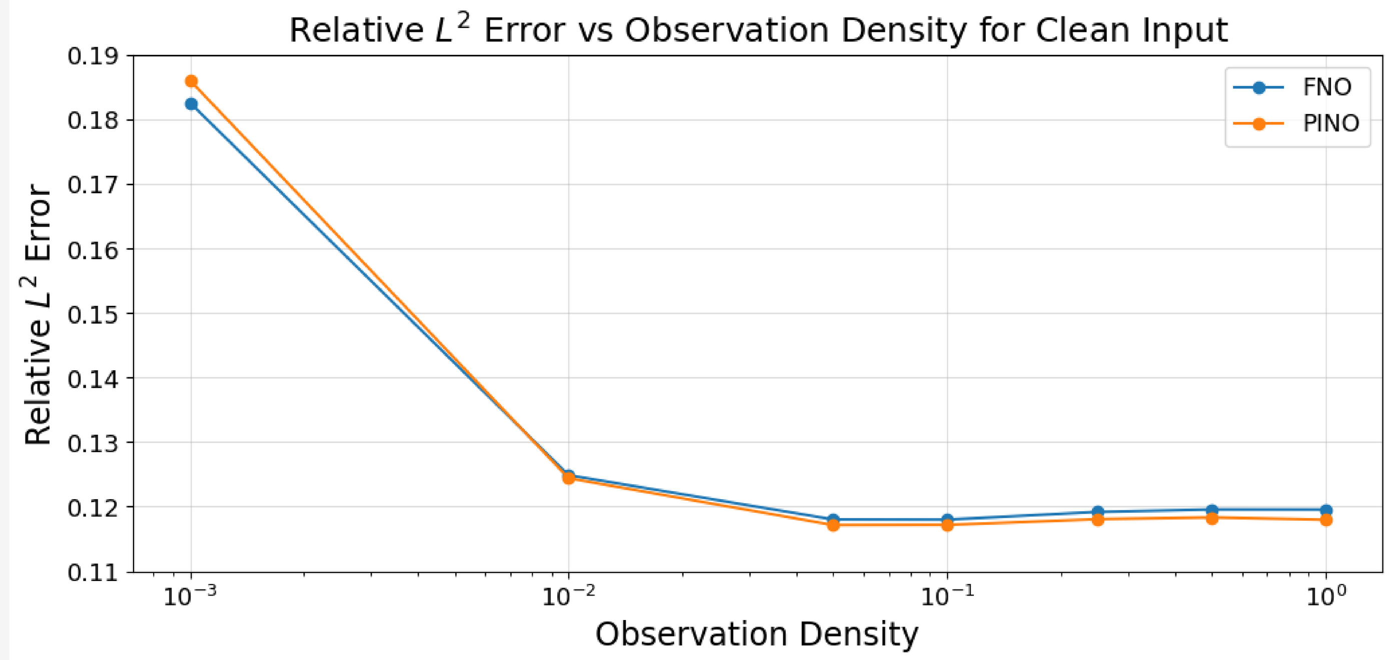
The input pressure field is corrupted with artificial Gaussian noise,

$$\tilde{u} = u + \sigma\,\mathrm{std}(u)\eta, \qquad \eta \sim \mathcal{N}(0,1), \qquad \sigma = 0.5.$$



Clean-trained FNO becomes unstable under noisy pressure inputs, while PINO is more stable. Noise-augmented FNO/PINO preserve the dominant permeability interface most consistently, showing that direct exposure to noisy inputs gives the strongest robustness benefit.

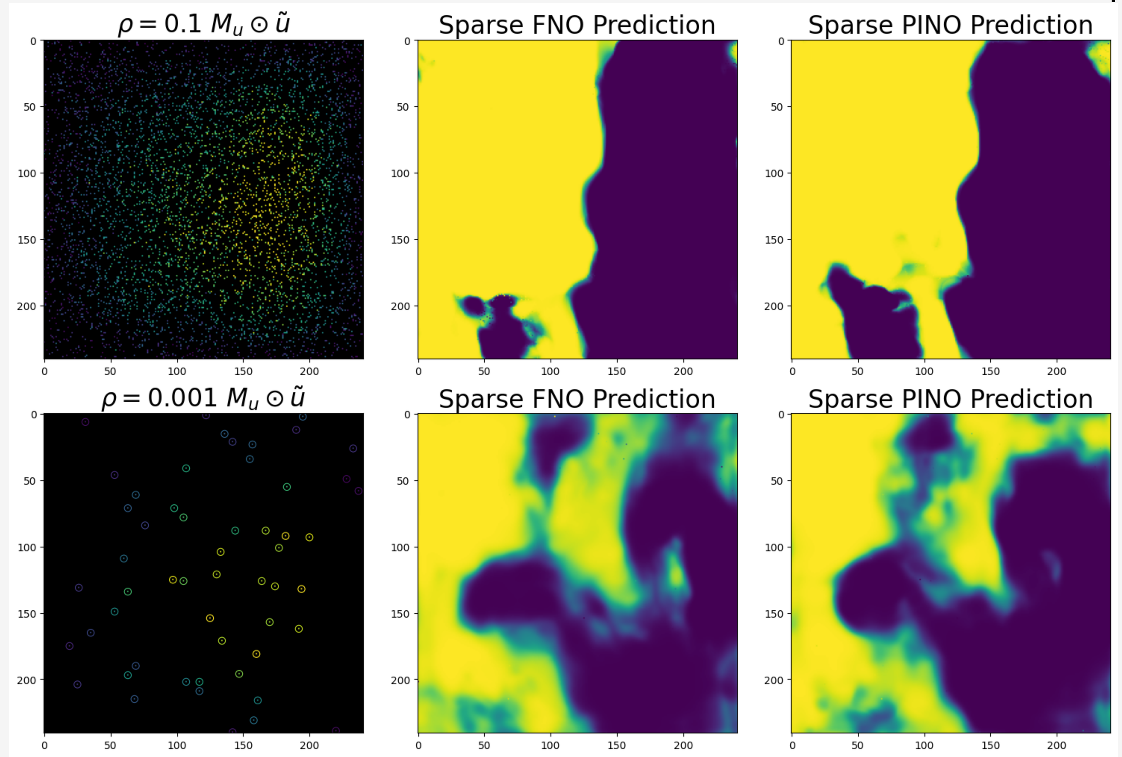
Sparse Observation Result 10



Sparse-trained FNO and PINO remain close to relative $L^2 \approx 0.12$ down to $\rho = 0.05$. Performance only deteriorates strongly at extremely low observation density, such as $\rho = 0.001$.

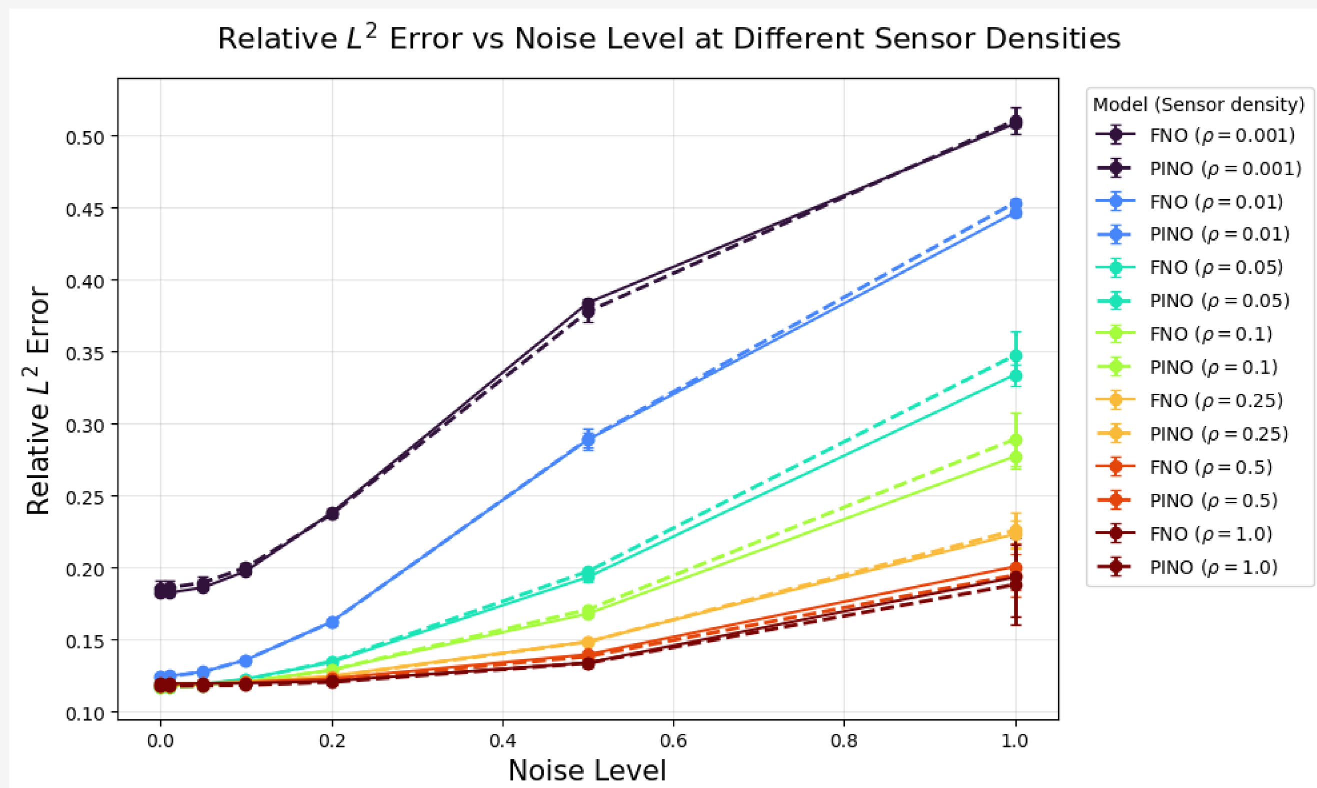
Sparse + Noisy Reconstruction 11

Using the same noisy input $\tilde{u}$ shown in block **9**, only a fraction ρ of grid points is observed. This evaluates the combined effect of noise and sparsity.



Sparse-trained models recover the dominant permeability regions, but sharp interfaces and small inclusions become less reliable as sparsity and noise increase.

Sparse + Noisy Observations 12



Noise has limited effect when the observation density is high, but reconstruction error increases strongly when both the noise level σ is large and the observation density ρ is low.

Answer to the Research Question 14

Accuracy: PDE-residual training does not universally reduce reconstruction error.

Physical consistency: A lower PDE residual does not necessarily imply a more accurate permeability reconstruction.

Robustness: PINO is more stable than clean-trained FNO under noise, but the strongest robustness comes from matching training to the test condition, especially noise augmentation and mask-aware sparse training.

Key Message 14

Physics-informed training acts mainly as a **stabilizing regularizer**, not as a replacement for training on the actual observation regime.

$$\textbf{robustness} \approx \text{input representation} + \text{training regime} + \text{data prior}.$$

Limitations and Future Work 15

- The benchmark uses restricted binary permeability fields, giving the models a strong data prior.
- Sparse observations are simulated with random grid masks, not real sensor layouts.
- Physical consistency is evaluated through the PDE residual; future work should include a forward-solve consistency check.
- Sparse and noisy inverse problems may be non-unique, motivating uncertainty-aware reconstructions.

References / Code 16

Z. Li et al., *Fourier Neural Operator for Parametric PDEs*, 2021.
Z. Li et al., *Physics-Informed Neural Operator*, 2023.
NVIDIA PhysicsNeMo framework.

Code:
`github.com/c-tallen/inverse_problems_with_pino`