

Background: Linear algebra and abstraction

Visual representations are powerful tools for communicating mathematical ideas. They can **aid the abstraction process** by coordinating visual and symbolic thinking [1]. However, concrete visual examples can also **hinder that process** by anchoring learners to specific instances [2].

Linear algebra is a particularly relevant case for examining this tension as it **requires a shift from concrete geometry to formal abstraction**. Students generally have **difficulty** with this shift to abstraction in linear algebra [3].

In response, textbooks may **reduce the number of illustrations in chapters on abstract spaces** [4, 5]. Whether this is the only effective strategy, or if illustrations could instead play a meaningful role in the transition, is the open question that motivates this study.

What is the **role of illustrations** across treatments of \mathbb{R}^2 - \mathbb{R}^3 , general \mathbb{R}^n , and abstract vector spaces within linear algebra textbooks?

Prior literature

Hillel [6] distinguishes three “modes of description”: the **geometric** (\mathbb{R}^2 - \mathbb{R}^3), the **algebraic** (\mathbb{R}^n), and the **abstract**. **Harel** [7] describes a set of “varieties” in which geometric models support learning (for example, the “metaphorical representation” variety). These works were adapted into the codebook.

Gueudet-Chartier [4] analyzed linear algebra textbooks, focusing on the pedagogical utility of the geometric model. Illustrations almost fully disappear in the transition to abstraction. **Kaneko and Takato** [5] supported this pattern. These studies establish a foundation for analyzing textbooks across different mathematical spaces.

The present study shifts the focus to the communicative function of the illustrations themselves.

Methodology

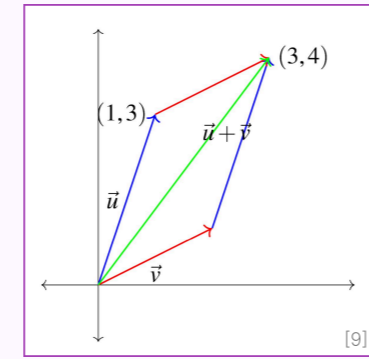
The study performs **thematic analysis** through **open coding** of illustrations from four textbooks [8-11], selected based on their relevance to tertiary education and attention to vector space theory.

Inductively, codes were developed from the data with **descriptive labels** for each illustration. **Deductively**, the process was informed by both **Hillel's** [6] modes of description and **Harel's** [7] typology.

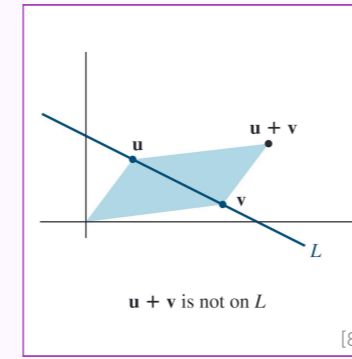
Relevant illustrations in chapters on vector spaces were labelled using this codebook, from which recurring patterns were identified.

Theme #1: De-spatialization as a bridge to abstraction

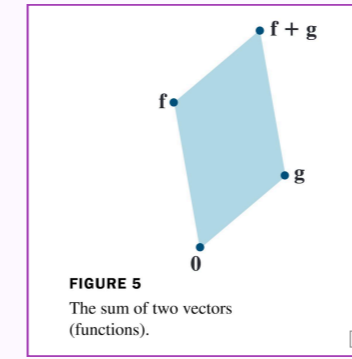
Spatial characteristics of illustrations are weakened in the transition to abstract spaces to help readers move away from geometric interpretation. We refer to this weakening as “de-spatialization”.



- a) \mathbb{R}^2 space
- Vectors as **arrows**: inherently spatial
 - **Cartesian axes**: represent space



- b) \mathbb{R}^n space
- Vectors as **points**: free from spatiality (magnitude and direction)



- c) Abstract space
- **Omits Cartesian axes**: concept of space no longer relevant

De-spatialization is also supported by **preferring 2D illustrations over 3D ones**. 2D images **avoid visual clutter** like perspective and foreshortening, putting more emphasis on the core of the concept.

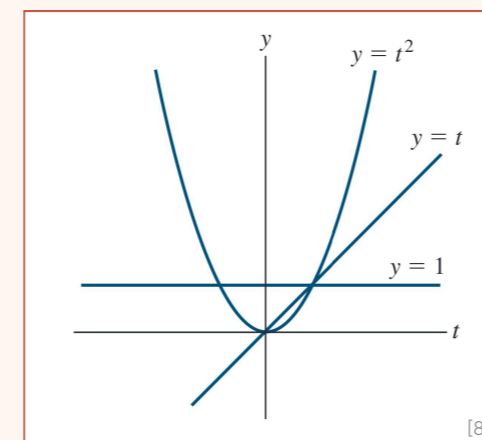
Theme #2: Coordination of multiple representations

Illustrations coordinate multiple forms of the same concept, making relationships between (seemingly unrelated) representations visible.

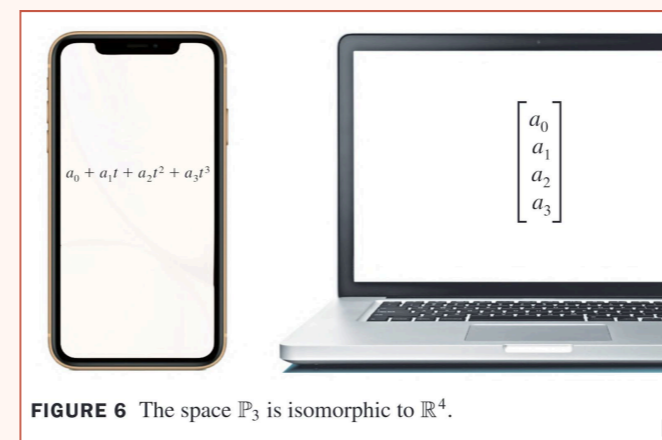
\mathbb{R}^2 - \mathbb{R}^3 : **Switching is literal**. Illustrations provide one-to-one translations between algebraic notation and geometric forms.

\mathbb{R}^n : **Switching is semi-literal**. Illustrations cannot literally represent higher-dimensional spaces, so they rely on lower-dimensional (2D/3D) stand-ins instead.

Abstract: **Switching is done through proxy visualizations**.



- a) By showing individual vectors (polynomials), rather than the space itself (polynomial space).



- b) By using a metaphorical model (everyday analogy of paired digital screens).

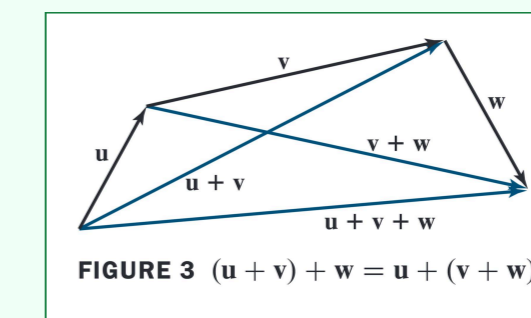
Metaphorical models make the link between symbolic and visual representations **weaker**, because the reader has to **abstract away non-literal elements**.

Theme #3: Contribution to theory architecture

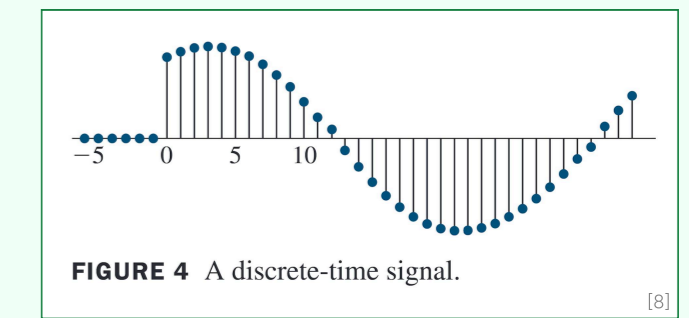
Illustrations help with the organization of the theory. Concept-defining ones help introduce new theory, while concept-stabilizing ones revisit established concepts to reinforce them.

Concept-defining illustrations are **prominent** in \mathbb{R}^2 - \mathbb{R}^3 . These function as the concept's **dominant visual reference**, potentially shaping understanding of subsequent theoretical developments.

Yet, they are **rare** in abstract contexts. Abstract spaces **rely heavily on concept-stabilizing** images. For example, the introduction of vector space axioms lacks a concept-defining illustration, and is instead stabilized in two settings.



- a) Space of arrows



- a) Space of sequences

A possible explanation is that abstract spaces are **too diverse** to be introduced through a single shared visual model. As a result, abstract spaces **may not develop a dominant visual reference**.

Discussion: Pedagogical implications

Because students struggle to abandon geometric intuition in the transition to abstraction [12], non-geometric metaphors may support visualization while **loosening dependence** on Euclidean space. Such metaphors may also prove useful for the development of visual, non-geometric models **specific to linear algebra**, which Gueudet-Chartier calls for [5].

Furthermore, concept-defining illustrations that reduce the general concept to a specific geometric instantiation may **confine** students to that geometric setting, which is a risk noted by prior research [5, 7].

These claims remain interpretive; future classroom observations are needed to determine how students actually experience these tensions.

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