The Fundamental Group of the Circle in Homotopy Type Theory

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Research question

- Does computer-checking the contents of the published paper: "Calculating the Fundamental Group of the Circle in Homotopy Type Theory"[2] within the Unimath library of Coq, confirm the results discussed therein?

The goal of the research paper by Mr Licata and Mr Shulman, was to formalize that the fundamental group of the circle is isomporphic to Z, which was done utilizing the Agda programming language; the topic of this research project is to translate this proof to Coq and the Unimath library.

Background

- If two paths can be continuously deformed into one another, we say that there exists a homotopy between these two paths.
- Paths with a homotopy between them belong to the same equivalence class under the relation of homotopy.

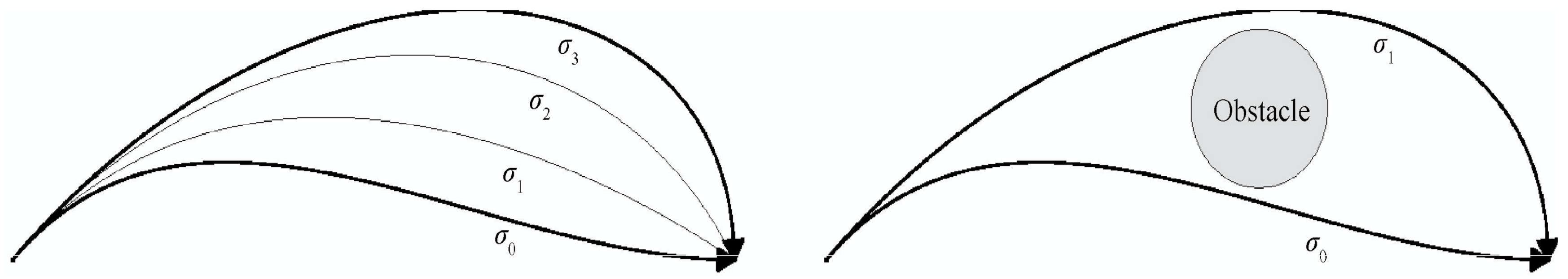


Figure 1. A homotopy of paths, the left image corresponds to a single homotopy class while the one on the right corresponds to distinct classes [4]

- A loop is a path with the same start- and endpoint.
- The **fundamental group of a space** is the group of homotopy classes of the loops contained in the space relative to some basepoint x_0, denoted as $\pi_1(X, x_0)$, with path composition as the group operator.
- The fundamental group of a space records information about that space, and can be used to tell spaces apart as well as give information about the basic shape, or holes, of the space. [3]
- The n-dimensional sphere, or simply n-sphere, is a space with a hole in the center. As an example the n-sphere S^1 can be viewed as a circle in two-dimensional space, and S^2 as a sphere in three-dimensions. The n-sphere can be given by the following equation:

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \ ||x|| = 1\}$$

- A group isomorphism can be defined as follows: Let $(G, *)$ and $(H, \circ)$ be groups, then:
  - A group homomorphism f : G → H is a function such that for all $x, y \in G$ we have:

$$f(x * y) = f(x) \circ f(y)$$

  - A group isomorphism is a group homomorphism which is a bijection, and thus represents a possible invertible mapping without loss of information
- Thus the mapping showing that the fundamental group of the circle is isomorphic to the integers is one of path composition to addition.

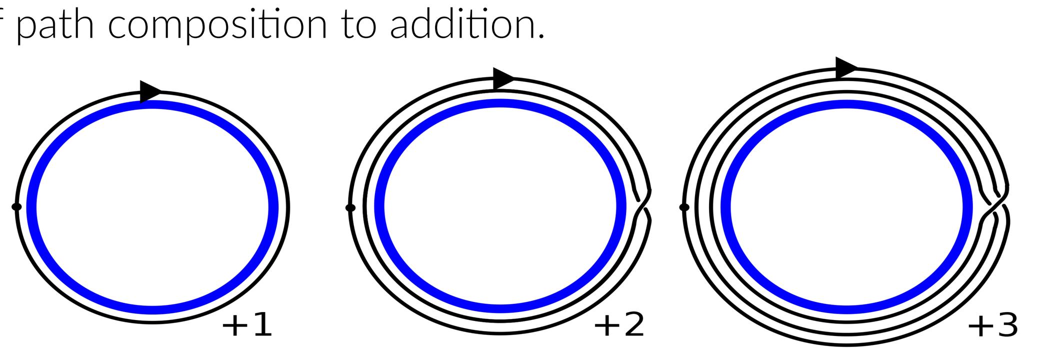


Figure 2. Loops around the circle; because the center contains a hole the loops cannot be deformed unto eachother. Thus, looping once, twice and thrice belong to separate homotopy classes. [1]

Higher homotopy groups

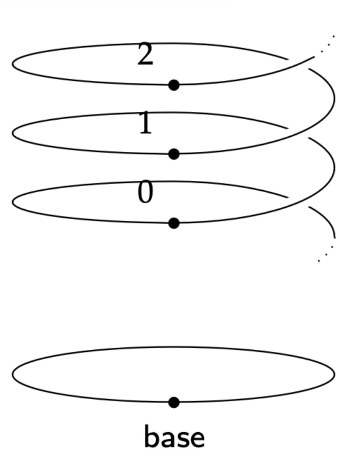
- The fundamental group of a space $\pi_1(X, x_0)$ is the first in a series of *homotopy groups* that provide additional information about a space.
  - For, example $\pi_2(X, x_0)$ provides additional information about the two-dimensional structure of the space X, and $\pi_3(X, x_0)$ about the 3-dimensional structure and so on. [3]
- $\pi_1(S^2)$ can be regarded as the mapping of the circle to the 2-sphere, however as the circle can be deformed "around the sphere" the homotopy class is trivial. Whereas, $\pi_2(S^2)$ can be regarded as the mapping of the 2-sphere to the 2-sphere, and is isomorphic to $\mathbb{Z}$ with the hole in the center preventing the kind the same kind of deforming.

	S^0	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8
π_1	0	$\mathbb{Z}$	0	0	0	0	0	0	0
π_2	0	0	$\mathbb{Z}$	0	0	0	0	0	0
π_3	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
π_4	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
π_5	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
π_6	0	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
π_7	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
π_8	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
π_9	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_{10}	0	0	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$
π_{11}	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$
π_{12}	0	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	0	0
π_{13}	0	0	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_2$	0

Figure 3. "Homotopy groups of spheres, the k^{th} homotopy group π_k of the n-sphere S^n is isomorphic to the group listed in each entry, with $\mathbb{Z}$ is the additive group of integers, and $\mathbb{Z}_m$ the cyclic group of order m". [3]

Universal Cover of the Circle

- The proof that was performed to show that the fundamental group of the circle is isomorphic to $\mathbb{Z}$ can be seen as the type-theoretic version of a proof in classical homotopy theory, through the use of *covering spaces*.
- In the figure to the right[3], the covering space of the circle S^1 can be represented as a helix projecting down unto the circle.
- One can map each point on the circle to a corresponding level on the helix, with the level determined by the loop concatenations that were performed. This mapping is called a *fibration*, and the fiber (on the helix) over each point on the circle is isomorphic to the integers. This fibration is called the universal cover of the circle.
- If you lift a loop that goes counterclockwise around the circle, you ascend one level in the helix, incrementing the integer that corresponds to the fiber. Going clockwise, descends a level on the helix, and thus decrementing the corresponding integer.



Showing that S^1 is isomorphic to $\mathbb{Z}$

- In Homotopy Type theory if you want to show an isomorphism between two types (in this case Circle and the Integers), one has to construct a *homotopy equivalence*. Which, in a classical homotopical sense, may be defined as follows:
  - A function f : $A \to B$ is an isomorphism if there is a function g : $B \to A$ such that both composites f ∘ g and g ∘ f are pointwise equal to the identitiy, i.e. such that $f \circ g \sim id_B$ and $g \circ f \sim id_A$.[3]
- In a somewhat oversimplified manner, the proofs that were performed in Coq are as follows, with each proof corresponding to the 'f', 'g', '$f \circ g$' and '$g \circ f$' functions as described in the definition above:
  - A proof *encode* which maps some composition of loops to the corresponding level on the helix.
  - A proof *decode* which maps a level on the helix to the composition of loops on the circle that correspond to it.
  - A proof *decode_encode* which shows that pre-composing with encode and post-composing with decode returns the composition of loops that were supplied to encode.
  - A proof *encode_intToLoop* which shows that the composition of loops constructed by the intToLoop function for some integer n, gets mapped to level of the helix that corresponds to n, by the encode function.
- Taking these proofs together a homotopy equivalence can be constructed, showing that S^1 is isomorphic to $\mathbb{Z}$.

Conclusions and Future work

- The research project demonstrated that there is a homotopy equivalence between S^1 and the Integers, establishing that the loop space of S^1 is equivalent to the Integers.
- Two proofs, however remain to be formalized:
  - The preserves_composition proof which identifies the fundamental group of S^1 with Int as a *group*, and shows that this equivalence is an isomorphism taking path composition to addition.
  - The encode_decode proof showing an equivalence between the Cover and the circle for general x of S^1.
- The Coq programming language and Unimath library do not possess support for the creation of Higher Inductive Types (HITs), requiring the use of axioms to replicate the correct behaviour for the loop.

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